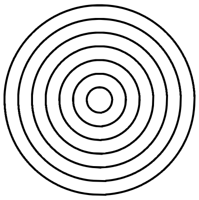
Deriving the Moment of Inertia of a Solid Sphere



Think about what shape can be added together to make a sphere and you should quickly realize it is a hollow sphere. Also taking the derivative of the volume of a sphere, $d(4/3)\pi r^3 = 4\pi r^2 dr$, which is the area of a hollow sphere times a small thickness. This is where the math lead us. (And sorry for the 2D "sphere".)

$$I = \int_0^R r^2 dm = \int_0^R r^2 \rho dV = \int_0^R r^2 \rho 4\pi r^2 dr = 4\pi\rho \int_0^R r^4 dr$$

$$I = 4\pi\rho \left(\frac{r^5}{5} \right) \Bigg|_0^R = \frac{4\pi\rho}{5}(R^5 - 0) = \frac{4\pi\rho R^5}{5} \text{ and } \rho = \frac{M}{(4/3)\pi R^3}$$

$$I = \frac{4\pi R^5}{5} \left(\frac{3M}{4\pi R^3} \right) = (3/5)MR^2$$

Whoa!!! That's not right! $I_{solid\ sphere} = (2/5)MR^2$. What went wrong?

Turns out it was our very first step: $I = \int r^2 dm$. This assumes that r is a constant for all of the *dm's* that we will sum up (integrate), but that is not so. For a hollow sphere not all the *dm*'s are r away from the pivot. (Full disclosure, this took me a while to figure out.)

However, we already have the "factor" for the mass that isn't r from the center of the sphere: 2/3. The moment of inertia of a hollow sphere is $(2/3)MR^2$. So let's try to add up a bunch of infinitely thin hollow spheres each with an infinitely small mass of *dm*.

$$I = \int_0^R (2/3) r^2 dm = (2/3) \int_0^R r^2 \rho dV = (2/3) \int_0^R r^2 \rho 4\pi r^2 dr = (8/3)\pi\rho \int_0^R r^4 dr$$

$$I = (8/3)\pi\rho \left(\frac{r^5}{5} \right) \Bigg|_0^R = \frac{8\pi\rho}{15}(R^5 - 0) = \frac{8\pi\rho R^5}{15} \text{ of course } \rho = \frac{M}{(4/3)\pi R^3}$$

$$I = \frac{8\pi R^5}{15} \left(\frac{3M}{4\pi R^3} \right) = (2/5)MR^2$$

Notes:

1) We learned here that just because a shape seems to be able to be summed up (integrated), you have to be careful. Just being aware of this may help in the future.

2) Most websites I went to integrated discs to make a solid sphere like I did with the hollow sphere. That seems more difficult than is necessary, in my opinion.